

# A Many-body Generalization of Quasi-solvable Models with Type C $\mathcal{N}$ -fold Supersymmetry (I) Regular Cases

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## Abstract

We make a generalization of the type C monomial space of a single variable, which was introduced in the construction of type C  $\mathcal{N}$ -fold supersymmetry, to several variables. Then, we construct the most general quasi-solvable second-order operators preserving this multivariate type C space. These operators of several variables are characterized by the fact that two different polynomial type solutions are available. In particular, we investigate and classify all the possible Schrödinger operators realized as a subclass of this family. It turns out that the rational, hyperbolic, and trigonometric Calogero-Sutherland models as well as some particular type of the elliptic Inozemtsev system, all associated with the  $BC_M$  root system, fall within the class.

*Key words:* quantum many-body problem, quasi(-exact) solvability, Calogero-Sutherland models, Inozemtsev models, supersymmetry

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## 1 Introduction

Finding exact solutions for an equation describing a physical system is one of the most important issues in mathematical physics. For quantum mechanical systems, the discovery of the underlying Lie-algebraic structure in Ref. [1] has stimulated systematic construction of (quasi-)solvable models. The first stage of the generalization of the idea to quantum many-body systems was however confronted with the difficulty due to the fact that second-order differential

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operators are in general not equivalent to Schrödinger operators, see, e.g., Refs. [2,3,4,5]. The next stage began with the success [6] of an algebraization of the known exactly solvable Calogero-Sutherland (CS) models [7,8]. The key ingredient in Ref. [6] is to construct the generators of  $\mathfrak{sl}(M+1)$  Lie algebra in terms of the elementary symmetric polynomials which reflect the permutation symmetry of the CS models. The similar methods were employed to show the (quasi-)solvability of the variety of the CS models [9,10,11,12]. Finally, the complete classification of the models constructed from the  $\mathfrak{sl}(M+1)$  algebraization was given in Ref. [13]. It turns out that all the obtained models are of the Inozemtsev type, which was shown to be classically integrable [14,15,16]. One of the key observations in Ref. [13] is the significance of the underlying symmetry of the solvable sector, namely, the subspace preserved by the Hamiltonian. Another aspect of the significant role of the symmetry in the one-body case was reported in Ref. [17] in connection with  $\mathcal{N}$ -fold supersymmetry.

In a previous paper [18], we developed a systematic algorithm to construct an  $\mathcal{N}$ -fold supersymmetric system starting from a function space preserved by one of the pair Hamiltonians. We then applied it to monomial type spaces to find out a new family of  $\mathcal{N}$ -fold supersymmetry called type C. As a byproduct, we found that for all finite values of the spectral parameter the characteristic equation of the Hamiltonians with type C  $\mathcal{N}$ -fold supersymmetry possesses two linearly independent series solutions around the origin in a suitable variable (cf. Section 5 in Ref. [18]). In this respect, there is a famous mathematical theorem, called Fuchs's theorem, which tells a sufficient condition for second-order linear differential operators of a single variable to acquire two linearly independent series solutions. The applicability of the theorem however is relatively limited and a generalization to partial differential equations is in general difficult. Therefore, the procedure presented in Ref. [18] has a potential possibility to be quite efficient in investigating what kind of linear differential operators, especially in several variables, can have two or more linearly independent series or polynomial solutions. In this context, it is natural to inquire in particular whether a similar approach could be applied to investigate what types of quantum many-body Hamiltonians admit the above property.

In this article, we will show that we can indeed systematically construct a family of quantum many-body systems with this characteristic by a suitable generalization of the one-body case in Ref. [18]. A key ingredient is how to generalize the so called *type C monomial space* of a single variable, which is preserved by the type C  $\mathcal{N}$ -fold supersymmetric Hamiltonians, to several variables. This is actually non-trivial problem because there are so many possible generalizations and we can hardly know *a priori* which one is preferable to others. For this purpose, we employ significant properties of the type C space of a single variable discussed in Ref. [18] as criteria. It turns out that the criteria really work advantageously in several contexts. In particular, we can mostly avoid the problem caused from the fact that second-order differential

operators are in general not equivalent to Schrödinger operators, and we can completely keep the same classification scheme as that in the single-variable case.

The article is organized as follows. In the next section, we summarize some definitions related to quasi-solvability in order to avoid ambiguity. In Section 3, we show how to generalize the type C monomial space of a single variable to several variables. Then in Section 4 we construct the most general second-order linear differential operator which preserves the generalized type C space. The obtained operators are however not equivalent to a Schrödinger operator in general. Thus we present in Section 5 a systematic procedure to extract the operators which are equivalent to a Schrödinger operator, namely, gauged Hamiltonians from the previously obtained family of the operators. In Section 6, we clarify how the gauged Hamiltonians obtained in Section 5 are embedded in the general type C operators by identifying the relation between the free parameters in the operators. Utilizing the form-invariance under the projective transformations, we fully classify the possible Hamiltonians preserving the generalized type C space in Section 7. The explicit form of the potential as well as the solvable sectors are exhibited in each case. In Section 8, we briefly discuss possibility and difficulty in the generalization of the type B monomial space to several variables. Finally, we summarize the results obtained here and discuss further possible developments of the present work in Section 9. Some useful formulas are summarized in Appendix.

## 2 Definition

First of all, we shall give the definition of quasi-solvability and some notions of its special cases based on Refs. [4,13]. A linear differential operator  $H$  of several variables  $q = (q_1, \dots, q_M)$  is said to be *quasi-solvable* if it preserves a finite dimensional functional space  $\mathcal{V}_{\mathcal{N}}$  whose basis admits an explicit analytic form  $\phi_i(q)$ :

$$H\mathcal{V}_{\mathcal{N}} \subset \mathcal{V}_{\mathcal{N}}, \quad \dim \mathcal{V}_{\mathcal{N}} = n(\mathcal{N}) < \infty, \quad \mathcal{V}_{\mathcal{N}} = \langle \phi_1(q), \dots, \phi_{n(\mathcal{N})}(q) \rangle. \quad (2.1)$$

An immediate consequence of the above definition of quasi-solvability is that, since we can calculate finite dimensional matrix elements  $\mathbf{S}_{k,l}$  defined by,

$$H\phi_k = \sum_{l=1}^{n(\mathcal{N})} \mathbf{S}_{k,l} \phi_l \quad \left( k = 1, \dots, n(\mathcal{N}) \right), \quad (2.2)$$

we can diagonalize the operator  $H$  and obtain its spectra in the space  $\mathcal{V}_{\mathcal{N}}$ , at least, algebraically. Furthermore, if the space  $\mathcal{V}_{\mathcal{N}}$  is a subspace of a Hilbert space  $L^2(S)$  ( $S \subset \mathbb{R}^M$ ) on which the operator  $H$  is naturally defined, the

calculable spectra and the corresponding vectors in  $\mathcal{V}_{\mathcal{N}}$  give the *exact* eigenvalues and eigenfunctions of  $H$ , respectively. In this case, the operator  $H$  is said to be *quasi-exactly solvable* (on  $S$ ). Otherwise, the calculable spectra and the corresponding vectors in  $\mathcal{V}_{\mathcal{N}}$  only give *local* solutions of the characteristic equation of  $H$ .

A quasi-solvable operator  $H$  of several variables is said to be *solvable* if it preserves an infinite flag of finite dimensional functional spaces  $\mathcal{V}_{\mathcal{N}}$ ,

$$\mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{V}_{\mathcal{N}} \subset \cdots, \quad (2.3)$$

whose bases admit explicit analytic forms, that is,

$$H\mathcal{V}_{\mathcal{N}} \subset \mathcal{V}_{\mathcal{N}}, \quad \dim \mathcal{V}_{\mathcal{N}} = n(\mathcal{N}) < \infty, \quad \mathcal{V}_{\mathcal{N}} = \langle \phi_1(q), \dots, \phi_{n(\mathcal{N})}(q) \rangle, \quad (2.4)$$

for  $\mathcal{N} = 1, 2, 3, \dots$ . Furthermore, if the sequence of the spaces (2.3) defined on  $S \subset \mathbb{R}^M$  satisfies,

$$\overline{\mathcal{V}_{\mathcal{N}}(S)} \rightarrow L^2(S) \quad (\mathcal{N} \rightarrow \infty), \quad (2.5)$$

the operator  $H$  is said to be *exactly solvable* (on  $S$ ).

### 3 A Generalization of the Type C Monomial Space

In this article, we shall consider a quantum system of  $M$  identical particles on a line. The Hamiltonian is thus given by

$$H = -\frac{1}{2} \sum_{i=1}^M \frac{\partial^2}{\partial q_i^2} + V(q_1, \dots, q_M), \quad (3.1)$$

where the potential has permutation symmetry:

$$V(\dots, q_i, \dots, q_j, \dots) = V(\dots, q_j, \dots, q_i, \dots) \quad \forall i \neq j. \quad (3.2)$$

To construct a quasi-solvable operator of the form (3.1), we shall follow the three steps after Ref. [13], namely, i) a *gauge* transformation on the Hamiltonian (3.1):

$$\tilde{H} = e^{\mathcal{W}(q)} H e^{-\mathcal{W}(q)}, \quad (3.3)$$

ii) a change of variables from  $q_i$  to  $z_i$  by a function  $z$  of a single variable:

$$z_i = z(q_i), \quad (3.4)$$

and iii) the introduction of the elementary symmetric polynomials of  $z_i$  defined by,

$$\sigma_k(z) = \sum_{i_1 < \dots < i_k}^M z_{i_1} \dots z_{i_k} \quad (k = 1, \dots, M), \quad \sigma_0 \equiv 1. \quad (3.5)$$

Due to the permutation symmetry of the original Hamiltonian (3.1), the gauged Hamiltonian (3.3) can be completely expressed in terms of the elementary symmetric polynomials (3.5).

The next task is to choose a vector space to be preserved by the gauged Hamiltonian (3.3). The type C monomial space of a single variable  $z$  is defined by [18],

$$\tilde{\mathcal{V}}_{\mathcal{N}_1, \mathcal{N}_2}^{(C)} = \langle 1, z, \dots, z^{\mathcal{N}_1-1}, z^\lambda, z^{\lambda+1}, \dots, z^{\lambda+\mathcal{N}_2-1} \rangle, \quad \mathcal{N} = \mathcal{N}_1 + \mathcal{N}_2, \quad (3.6)$$

where  $\mathcal{N}_1$  and  $\mathcal{N}_2$  are positive integers satisfying  $\mathcal{N}_1 \geq \mathcal{N}_2$  and  $\lambda$  is a real number with the restriction

$$\lambda \in \mathbb{R} \setminus \{-\mathcal{N}_2, -\mathcal{N}_2 + 1, \dots, \mathcal{N}_1\}, \quad (3.7)$$

and with  $\lambda \neq -\mathcal{N}_2 - 1, \mathcal{N}_1 + 1$  if  $\mathcal{N}_1 = 1$  or  $\mathcal{N}_2 = 1$ . It is however evident that there are many possibilities in generalizing the space (3.6) to several variables. Hence, we need suitable principles in order to render the possible generalizations unique. For this purpose, we shall recall the characteristic features of the type C space (3.6) discussed in Ref. [18]. First of all, the space (3.6) decomposes as the direct sum of two spaces:

$$\tilde{\mathcal{V}}_{\mathcal{N}_1, \mathcal{N}_2}^{(C)} = \tilde{\mathcal{V}}_{\mathcal{N}_1}^{(A)} \oplus z^\lambda \tilde{\mathcal{V}}_{\mathcal{N}_2}^{(A)}, \quad (3.8)$$

where  $\tilde{\mathcal{V}}_{\mathcal{N}_k}^{(A)}$  is the type A monomial space of dimension  $\mathcal{N}_k$  defined by

$$\tilde{\mathcal{V}}_{\mathcal{N}_k}^{(A)} = \langle 1, z, \dots, z^{\mathcal{N}_k-1} \rangle. \quad (3.9)$$

Thus, we shall generalize the type C space (3.6) so that a similar decomposition is realized. The next task is then to find a several-variable counterpart of the type A space (3.9). In this respect, we note the fact that the type A space (3.9) provides an  $\mathfrak{sl}(2)$  module, which is the foundations of the  $\mathfrak{sl}(2)$  construction of quasi-solvable models in Ref. [1]. Therefore, a natural generalization of the type A space to several variables is provided by an  $\mathfrak{sl}(M+1)$  module given by

$$\tilde{\mathcal{V}}_{\mathcal{N}_k; M}^{(A)} = \left\langle \sigma_1^{n_1} \dots \sigma_M^{n_M} : n_i \in \mathbb{Z}_{\geq 0}, 0 \leq \sum_{i=1}^M n_i \leq \mathcal{N}_k - 1 \right\rangle. \quad (3.10)$$

Indeed, this space is exactly what the quasi-solvable  $M$ -body Hamiltonians constructed in Ref. [13] from  $\mathfrak{sl}(M+1)$  generators preserve, and reduces to the

type A space of a single variable (3.9) when  $M = 1$ . From the decomposition criterion, the generalized type C space we shall take thus have the following form:

$$\tilde{\mathcal{V}}_{\mathcal{N}_1, \mathcal{N}_2; M}^{(C)} = \tilde{\mathcal{V}}_{\mathcal{N}_1; M}^{(A)} \oplus f_\lambda(z) \tilde{\mathcal{V}}_{\mathcal{N}_2; M}^{(A)}. \quad (3.11)$$

The function  $f_\lambda$  in Eq. (3.11) must be  $f_\lambda(z) = z^\lambda$  when  $M = 1$  in order that the space (3.11) reduces to Eq. (3.8). Next, we recall the fact that the type C space (3.6) is form-invariant under a special projective transformation. More explicitly, the space (3.6) satisfies

$$\tilde{\mathcal{V}}_{\mathcal{N}_1, \mathcal{N}_2}^{(C)}[z, \lambda] \mapsto z^{\mathcal{N}_1-1} \tilde{\mathcal{V}}_{\mathcal{N}_1, \mathcal{N}_2}^{(C)}[z^{-1}, \lambda] = \tilde{\mathcal{V}}_{\mathcal{N}_1, \mathcal{N}_2}^{(C)}[z, \mathcal{N}_1 - \mathcal{N}_2 - \lambda]. \quad (3.12)$$

In Ref. [13] it was shown that the invariance under projective transformations of the vector space preserved by the Hamiltonians plays crucial roles in constructing quasi-solvable quantum many-body systems. Thus, we shall require that the generalized type C space has a similar property to Eq. (3.12). To see how the space of the form (3.11) behaves under the special projective transformation, we first note that the elementary symmetric polynomials (3.5) are transformed according to

$$\sigma_k(z) \mapsto \sigma_k(z^{-1}) = \sigma_{M-k}(z) \sigma_M(z)^{-1}. \quad (3.13)$$

From this formula, we easily see that the type A space of several variables defined by Eq. (3.10) has the following invariance property:

$$\tilde{\mathcal{V}}_{\mathcal{N}_k; M}^{(A)}[z] \mapsto \tilde{\mathcal{V}}_{\mathcal{N}_k; M}^{(A)}[z^{-1}] = \sigma_M^{-(\mathcal{N}_k-1)} \tilde{\mathcal{V}}_{\mathcal{N}_k; M}^{(A)}[z]. \quad (3.14)$$

Hence, the space (3.11) transforms as follows:

$$\begin{aligned} \tilde{\mathcal{V}}_{\mathcal{N}_1, \mathcal{N}_2; M}^{(C)}[z, \lambda] &\mapsto \tilde{\mathcal{V}}_{\mathcal{N}_1, \mathcal{N}_2; M}^{(C)}[z^{-1}, \lambda] \\ &= \sigma_M^{-(\mathcal{N}_1-1)} \left( \tilde{\mathcal{V}}_{\mathcal{N}_1; M}^{(A)}[z] \oplus \sigma_M^{\mathcal{N}_1-\mathcal{N}_2} f_\lambda(z^{-1}) \tilde{\mathcal{V}}_{\mathcal{N}_2; M}^{(A)}[z] \right). \end{aligned} \quad (3.15)$$

It is now clear that the space (3.11) has a suitable form-invariance similar to Eq. (3.12) if and only if the function  $f_\lambda$  in Eq. (3.11) satisfies

$$\sigma_M^{\mathcal{N}_1-\mathcal{N}_2} f_\lambda(z^{-1}) = f_{\mathcal{N}_1-\mathcal{N}_2-\lambda}(z). \quad (3.16)$$

Actually if this is the case, we have

$$\tilde{\mathcal{V}}_{\mathcal{N}_1, \mathcal{N}_2; M}^{(C)}[z, \lambda] \mapsto \sigma_M^{\mathcal{N}_1-1} \tilde{\mathcal{V}}_{\mathcal{N}_1, \mathcal{N}_2; M}^{(C)}[z^{-1}, \lambda] = \tilde{\mathcal{V}}_{\mathcal{N}_1, \mathcal{N}_2; M}^{(C)}[z, \mathcal{N}_1 - \mathcal{N}_2 - \lambda], \quad (3.17)$$

which reduces exactly to Eq. (3.12) when  $M = 1$ . The solution of Eq. (3.16) is easily found if we note that the dependence on  $\mathcal{N}_1 - \mathcal{N}_2$  only comes from the variable  $\sigma_M$  and take the formula (3.13) into account. Only the possible solution for  $f_\lambda$ , which reduces to  $z^\lambda$  when  $M = 1$ , is given by

$$f_\lambda(z) = \sigma_M(z)^\lambda. \quad (3.18)$$

In summary, we have generalized the type C monomial space (3.6) of a single variable to several variables by the requirement of the decomposition structure (3.11) and of the form-invariance under the special projective transformation (3.17), and finally obtain

$$\tilde{\mathcal{V}}_{\mathcal{N}_1, \mathcal{N}_2; M}^{(C)} = \tilde{\mathcal{V}}_{\mathcal{N}_1; M}^{(A)} \oplus \sigma_M^\lambda \tilde{\mathcal{V}}_{\mathcal{N}_2; M}^{(A)}. \quad (3.19)$$

#### 4 Construction of Quasi-solvable Operators

In this section, we shall construct the general quasi-solvable operators of (at most) second-order leaving the type C space (3.19) invariant. For the first-order differential operators, we have the following set of independent ones:

$$\frac{\partial}{\partial \sigma_{\bar{i}}} \equiv E_{0\bar{i}} \quad (\bar{i} = 1, \dots, M-1), \quad (4.1a)$$

$$\sigma_i \frac{\partial}{\partial \sigma_{\bar{j}}} \equiv E_{i\bar{j}} \quad (i = 1, \dots, M; \bar{j} = 1, \dots, M-1), \quad (4.1b)$$

$$\sigma_M \frac{\partial}{\partial \sigma_M} \equiv E_{MM}. \quad (4.1c)$$

In the above and hereafter, we employ the convention that the Latin indices take integer values from 1 to  $M$  while the ones with bar take integer values from 1 to  $M-1$ . We note that  $E_{0M}$  and  $E_{\bar{i}M}$  cannot preserve the elements  $\sigma_1^{n_1} \dots \sigma_{M-1}^{n_{M-1}} \sigma_M^\lambda$  inside the space  $\tilde{\mathcal{V}}_{\mathcal{N}_1, \mathcal{N}_2; M}^{(C)}$  due to the restriction on  $\lambda$ , Eq. (3.7). We also note that any first-order operator which raises the degree cannot preserve the space  $\tilde{\mathcal{V}}_{\mathcal{N}_1, \mathcal{N}_2; M}^{(C)}$ .

For the independent second-order differential operators, we have the following set:

$$\frac{\partial^2}{\partial \sigma_{\bar{k}} \partial \sigma_{\bar{l}}} = E_{0\bar{k}} E_{0\bar{l}} \quad (\bar{k} \geq \bar{l}), \quad (4.2)$$

$$\sigma_i \frac{\partial^2}{\partial \sigma_{\bar{k}} \partial \sigma_{\bar{l}}} = E_{i\bar{k}} E_{0\bar{l}} \quad (\bar{k} \geq \bar{l}), \quad (4.3a)$$

$$\sigma_M \frac{\partial^2}{\partial \sigma_M \partial \sigma_{\bar{k}}} = E_{MM} E_{0\bar{k}}, \quad (4.3b)$$

$$\frac{\partial}{\partial \sigma_M} \left( \sigma_M \frac{\partial}{\partial \sigma_M} - \lambda \right) \equiv E_{MM, 0M}, \quad (4.3c)$$

$$\sigma_i \sigma_j \frac{\partial^2}{\partial \sigma_{\bar{k}} \partial \sigma_{\bar{l}}} = E_{i\bar{k}} E_{j\bar{l}} - \delta_{j\bar{k}} E_{i\bar{l}} \quad (i \geq j; \bar{k} \geq \bar{l}), \quad (4.4a)$$

$$\sigma_M \sigma_i \frac{\partial^2}{\partial \sigma_M \partial \sigma_{\bar{k}}} = E_{MM} E_{i\bar{k}} - \delta_{iM} E_{M\bar{k}}, \quad (4.4b)$$

$$\sigma_M^2 \frac{\partial^2}{\partial \sigma_M^2} = E_{MM} E_{MM} - E_{MM}, \quad (4.4c)$$

$$\sigma_i \frac{\partial}{\partial \sigma_M} \left( \sigma_M \frac{\partial}{\partial \sigma_M} - \lambda \right) \equiv E_{MM, \bar{i}M} \quad (4.4d)$$

$$\sigma_i \left( \mathcal{N}_1 - 1 - \sum_{k=1}^M \sigma_k \frac{\partial}{\partial \sigma_k} \right) \left( \lambda + \mathcal{N}_2 - 1 - \sum_{l=1}^M \sigma_l \frac{\partial}{\partial \sigma_l} \right) \equiv E_{i0,00}. \quad (4.5)$$

The operators (4.3c), (4.4d), and (4.5) cannot be expressed as polynomials of the set of the first-order operators (4.1). In the case of  $M = 1$ , only the four independent operators, namely, Eqs. (4.1c), (4.3c), (4.4c), and (4.5) with  $i = M = 1$  exist, which reproduces the result for the type C quasi-solvable operators of a single variable given in Ref. [18]. We also note the important fact that all the operators listed above preserve *separately* both the subspaces of  $\tilde{\mathcal{V}}_{\mathcal{N}_1, \mathcal{N}_2; M}^{(C)}$  in Eq. (3.19), as in the case of  $M = 1$  [18]. The most general linear differential operator of (at most) second-order which leaves the type C space (3.19) invariant is thus given by

$$\begin{aligned} \tilde{\mathcal{H}} = & - \sum_i A_{i0,00} E_{i0,00} - \sum_{\mu \geq \nu, \bar{k} \geq \bar{l}} A_{\mu\bar{k}, \nu\bar{l}} E_{\mu\bar{k}} E_{\nu\bar{l}} - \sum_{\bar{\mu}} A_{MM, \bar{\mu}M} E_{MM, \bar{\mu}M} \\ & - \sum_{\mu, \bar{k}} A_{MM, \mu\bar{k}} E_{MM} E_{\mu\bar{k}} - A_{MM, MM} E_{MM} E_{MM} \\ & + \sum_{\mu, \bar{k}} B_{\mu\bar{k}} E_{\mu\bar{k}} + B_{MM} E_{MM} - c_0, \end{aligned} \quad (4.6)$$

where the coefficients  $A$ ,  $B$  with indices and  $c_0$  are real constants. The Greek indices without(with) bar take integer values from 0 to  $M(M-1)$ , respectively. The summation above is understood to take all the possible integer values indicated by the corresponding type of indices. In terms of the variables  $\sigma_k$ , the operator  $\tilde{\mathcal{H}}$  is expressed as

$$\begin{aligned} \tilde{\mathcal{H}} = & - \sum_{\bar{k}, \bar{l}=1}^{M-1} \left[ \mathbf{A}_0(\sigma) \sigma_{\bar{k}} \sigma_{\bar{l}} + \mathbf{A}_{\bar{k}\bar{l}}(\sigma) \right] \frac{\partial^2}{\partial \sigma_{\bar{k}} \partial \sigma_{\bar{l}}} - \sum_{\bar{k}=1}^{M-1} \left[ 2\mathbf{A}_0(\sigma) \sigma_{\bar{k}} \right. \\ & \left. + \mathbf{A}_{M\bar{k}}(\sigma) \right] \sigma_M \frac{\partial^2}{\partial \sigma_M \partial \sigma_{\bar{k}}} - \left[ \mathbf{A}_0(\sigma) \sigma_M + \mathbf{A}_{MM}(\sigma) \right] \sigma_M \frac{\partial^2}{\partial \sigma_M^2} \\ & + \sum_{k=1}^M \left[ (\mathcal{N} + \lambda - 3) \mathbf{A}_0(\sigma) \sigma_k - \mathbf{B}_k(\sigma) \right] \frac{\partial}{\partial \sigma_k} - \mathbf{C}(\sigma), \end{aligned} \quad (4.7)$$

where  $\mathbf{A}_0$ ,  $\mathbf{A}_{kl}$ ,  $\mathbf{B}_k$ , and  $\mathbf{C}$  are polynomials of several variables of at most



second-degree given by

$$\mathbf{A}_0(\sigma) = \sum_{i=1}^M A_{i0,00} \sigma_i, \quad (4.8a)$$

$$\mathbf{A}_{\bar{k}\bar{l}}(\sigma) = \sum_{i \geq j}^M A_{i\bar{k},j\bar{l}} \sigma_i \sigma_j + \sum_{i=1}^M A_{i\bar{k},0\bar{l}} \sigma_i + A_{0\bar{k},0\bar{l}} \quad (\bar{k} \geq \bar{l}), \quad (4.8b)$$

$$\mathbf{A}_{Mk}(\sigma) = \sum_{i=1}^M A_{MM,ik} \sigma_i + A_{MM,0k}, \quad (4.8c)$$

$$\mathbf{B}_{\bar{k}}(\sigma) = \sum_{i=1}^M \left( \sum_{(i \geq) \bar{j} (\geq \bar{k})}^{M-1} A_{i\bar{j},j\bar{k}} - B_{i\bar{k}} \right) \sigma_i + A_{MM,M\bar{k}} \sigma_M - B_{0\bar{k}}, \quad (4.8d)$$

$$\mathbf{B}_M(\sigma) = -(\lambda - 1) \mathbf{A}_{MM}(\sigma) + (\lambda A_{MM,MM} - B_{MM}) \sigma_M, \quad (4.8e)$$

$$\mathbf{C}(\sigma) = (\mathcal{N}_1 - 1)(\mathcal{N}_2 + \lambda - 1) \mathbf{A}_0(\sigma) + c_0. \quad (4.8f)$$

Except for the operators  $E_{i0,00}$  given by Eq. (4.5), all the operators in Eq. (4.6) leave invariant both the subspaces in Eq. (3.19) for *arbitrary* positive-integer values of  $\mathcal{N}_1$  and  $\mathcal{N}_2$ . Therefore, the operator  $\tilde{\mathcal{H}}$  is solvable if all the coefficients  $A_{i0,00}$  vanish, or equivalently, if

$$\mathbf{A}_0(\sigma) = 0. \quad (4.9)$$

## 5 Extraction of Schrödinger Operators

In the preceding section, we have constructed the most general quasi-solvable second-order operator  $\tilde{H}$  preserving the type C space (3.19). By applying a similarity transformation on  $\tilde{H}$ , we may obtain a family of quasi-solvable operators. However, second-order linear differential operators of several variables are in general not equivalent to Schrödinger operators. This fact is one of the most obstacles in constructing a quasi-solvable *quantum* many-body systems. Recently in Refs. [13,19], it was shown that the amount of the difficulty can be significantly reduced by considering the underlying symmetry of the invariant space of quasi-solvable operators. In this section, we will show that the symmetry consideration is indeed quite efficient in our case too.

To begin with, let us briefly review the consequence in the case of the type A space (3.10) (see Ref. [19] for the details). The symmetry which transforms the type A space to itself is  $GL(2, \mathbb{R})$  of linear fractional transformations of  $z_i$  introduced by

$$z_i \mapsto \hat{z}_i = \frac{\alpha z_i + \beta}{\gamma z_i + \delta} \quad (\alpha, \beta, \gamma, \delta \in \mathbb{R}; \Delta \equiv \alpha\delta - \beta\gamma \neq 0). \quad (5.1)$$

Then, it is easy to see that the type A space (3.10) is invariant under the  $GL(2, \mathbb{R})$  transformation induced by (5.1):

$$\tilde{\mathcal{V}}_{\mathcal{N}_k}^{(A)}[z] \mapsto \prod_{i=1}^M (\gamma z_i + \delta)^{\mathcal{N}_k-1} \tilde{\mathcal{V}}_{\mathcal{N}_k}^{(A)}[\hat{z}] = \tilde{\mathcal{V}}_{\mathcal{N}_k}^{(A)}[z]. \quad (5.2)$$

From this invariance, we can conclude that any operator  $\tilde{\mathcal{H}}^{(A)}$  which preserves the type A space must be shape invariant under the  $GL(2, \mathbb{R})$  transformation as follows:

$$\begin{aligned} \tilde{\mathcal{H}}_{\mathcal{N}_k}^{(A)}[z, a_l] &\mapsto \prod_{i=1}^M (\gamma z_i + \delta)^{\mathcal{N}_k-1} \tilde{\mathcal{H}}_{\mathcal{N}_k}^{(A)}[\hat{z}, a_l] \prod_{i=1}^M (\gamma z_i + \delta)^{-(\mathcal{N}_k-1)} \\ &= \tilde{\mathcal{H}}_{\mathcal{N}_k}^{(A)}[z, \hat{a}_l], \end{aligned} \quad (5.3)$$

where  $a_l$  stands for all the free parameters involved in  $\tilde{\mathcal{H}}_{\mathcal{N}_k}^{(A)}$ . In particular, all the second-order operators  $\tilde{H}_{\mathcal{N}_k}^{(A)}$  which preserve the type A space and are equivalent to the Schrödinger operators as well must have the latter shape-invariance. From this requirement we can derive that  $\tilde{H}_{\mathcal{N}_k}^{(A)}$  must have the following form:

$$\tilde{H}_{\mathcal{N}_k}^{(A)} = \tilde{H}_{\mathcal{N}_k}^{(0)} - \tilde{H}_{\mathcal{N}_k}^{(1)} - 2c_k \tilde{H}_{\mathcal{N}_k}^{(2)} - R_k, \quad (5.4)$$

where  $c_k$  and  $R_k$  are constants and  $\tilde{H}_{\mathcal{N}_k}^{(j)}$  ( $j = 0, 1, 2$ ) are all shape-invariant operators satisfying Eq. (5.3) as the following:

$$\begin{aligned} \tilde{H}_{\mathcal{N}_k}^{(0)} &= - \sum_{i=1}^M O_4^{(k)}(z_i) \frac{\partial^2}{\partial z_i^2} + \frac{\mathcal{N}_k - 2}{2} \sum_{i=1}^M O_4^{(k)'}(z_i) \frac{\partial}{\partial z_i} \\ &\quad - \frac{(\mathcal{N}_k - 1)(\mathcal{N}_k - 2)}{12} \sum_{i=1}^M O_4^{(k)''}(z_i), \end{aligned} \quad (5.5)$$

$$\tilde{H}_{\mathcal{N}_k}^{(1)} = \sum_{i=1}^M O_2^{(k)}(z_i) \frac{\partial}{\partial z_i} - \frac{\mathcal{N}_k - 1}{2} \sum_{i=1}^M O_2^{(k)'}(z_i), \quad (5.6)$$

$$\begin{aligned} \tilde{H}_{\mathcal{N}_k}^{(2)} &= \sum_{i \neq j}^M \frac{O_4^{(k)}(z_i)}{z_i - z_j} \frac{\partial}{\partial z_i} - \frac{M - 1}{4} \sum_{i=1}^M O_4^{(k)'}(z_i) \frac{\partial}{\partial z_i} \\ &\quad - \frac{\mathcal{N}_k - 1}{4} \left[ \sum_{i \neq j}^M \frac{O_4^{(k)'}(z_i)}{z_i - z_j} - \frac{M - 1}{3} \sum_{i=1}^M O_4^{(k)''}(z_i) \right]. \end{aligned} \quad (5.7)$$

The functions  $O_n^{(k)}(z)$  ( $n = 2, 4$ ) in Eqs. (5.5)–(5.7) are polynomials of (at most)  $n$ th-degree and are transformed under the  $GL(2, \mathbb{R})$  transformation according to

$$O_n^{(k)}(z) \mapsto \Delta^{-n/2} (\gamma z + \delta)^n O_n^{(k)}(\hat{z}). \quad (5.8)$$

Keeping these facts in mind, let us come back to consider the type C case. As we have mentioned previously, all the operators in Eq. (4.6) and hence  $\tilde{H}$  preserve each of the subspaces in Eq. (3.19) separately. In other words, the second-order operator  $\sigma_M^{-(k-1)\lambda} \tilde{H} \sigma_M^{(k-1)\lambda}$  ( $k = 1, 2$ ) leaves the type A space  $\tilde{\mathcal{V}}_{\mathcal{N}_k}^{(A)}$  of dimension  $\mathcal{N}_k$  invariant. On the other hand, the results for the type A case just summarized tell us that these operators are equivalent to a Schrödinger operator if and only if they have the form given by Eqs. (5.4)–(5.7). Therefore, the general type C operator  $\tilde{H}$  (4.7) is equivalent to a Schrödinger operator of several variables if and only if the following conditions are simultaneously satisfied (up to constants):

$$\tilde{H} = \tilde{H}_{\mathcal{N}_1}^{(A)}, \quad \sigma_M^{-\lambda} \tilde{H} \sigma_M^{\lambda} = \tilde{H}_{\mathcal{N}_2}^{(A)}. \quad (5.9)$$

The explicit form of the operator  $\sigma_M^{\lambda} \tilde{H}_{\mathcal{N}_k}^{(A)} \sigma_M^{-\lambda}$  is calculated as

$$\begin{aligned} \sigma_M^{\lambda} \tilde{H}_{\mathcal{N}_k}^{(A)} \sigma_M^{-\lambda} = & - \sum_{i=1}^M O_4^{(k)}(z_i) \frac{\partial^2}{\partial z_i^2} + \sum_{i=1}^M \left[ \frac{\mathcal{N}_k - 2 + (M-1)c_k}{2} O_4^{(k)'}(z_i) \right. \\ & + 2\lambda O_4^{(k)}(z_i) z_i^{-1} - O_2^{(k)}(z_i) \left. \right] \frac{\partial}{\partial z_i} - 2c_k \sum_{i \neq j}^M \frac{O_4^{(k)}(z_i)}{z_i - z_j} \frac{\partial}{\partial z_i} \\ & + c_k \sum_{i \neq j}^M \left[ \frac{\mathcal{N}_k - 1}{2} O_4^{(k)'}(z_i) + 2\lambda O_4^{(k)}(z_i) z_i^{-1} \right] \frac{1}{z_i - z_j} \\ & - \sum_{i=1}^M \left[ \frac{\mathcal{N}_k - 1}{12} (\mathcal{N}_k - 2 + 2(M-1)c_k) O_4^{(k)''}(z_i) \right. \\ & + \frac{\mathcal{N}_k - 2 + (M-1)c_k}{2} \lambda O_4^{(k)'}(z_i) z_i^{-1} + \lambda(\lambda+1) O_4^{(k)}(z_i) z_i^{-2} \\ & \left. - \frac{\mathcal{N}_k - 1}{2} O_2^{(k)'}(z_i) - \lambda O_2^{(k)}(z_i) z_i^{-1} \right] - R_k. \quad (5.10) \end{aligned}$$

It is evident that the condition (5.9) is satisfied if and only if the coefficients of the differential operators of each order in  $\tilde{H}_{\mathcal{N}_1}^{(A)}$  and  $\sigma_M^{\lambda} \tilde{H}_{\mathcal{N}_2}^{(A)} \sigma_M^{-\lambda}$  coincide with each other (up to a constant for the zeroth-order). From Eq. (5.10) we thus obtain the following set of conditions:

$$O_4^{(1)}(z_i) = O_4^{(2)}(z_i) \equiv A(z_i), \quad (5.11)$$

$$O_2^{(2)}(z_i) - O_2^{(1)}(z_i) = \frac{\mathcal{N}_2 - \mathcal{N}_1}{2} A'(z_i) + 2\lambda A(z_i) z_i^{-1}, \quad (5.12)$$

$$\begin{aligned} & \sum_{i=1}^M \left[ \frac{\mathcal{N}_2 - 1}{2} O_2^{(2)'}(z_i) + \lambda O_2^{(2)}(z_i) z_i^{-1} - \frac{\mathcal{N}_1 - 1}{2} O_2^{(1)'}(z_i) \right] \\ & = \sum_{i=1}^M \left[ \frac{\mathcal{N}_2 - \mathcal{N}_1}{12} (\mathcal{N} - 3 + 2(M-1)c) A''(z_i) + \frac{\mathcal{N}_2 - 2 + (M-1)c}{2} \lambda A'(z_i) z_i^{-1} \right. \end{aligned}$$

$$+ \lambda(\lambda + 1)A(z_i)z_i^{-2} \Big] - c \sum_{i \neq j}^M \left[ \frac{\mathcal{N}_2 - \mathcal{N}_1}{2} A'(z_i) + 2\lambda A(z_i)z_i^{-1} \right] \frac{1}{z_i - z_j} + \text{const.}, \quad (5.13)$$

where  $c \equiv c_1 = c_2$ . By substituting Eq. (5.12) for  $O_2^{(2)}(z_i)$  in Eq. (5.13), the third condition can be expressed solely in terms of  $O_2^{(1)}(z_i)$  and  $A(z_i)$ :

$$\begin{aligned} & \sum_{i=1}^M \left[ \frac{\mathcal{N}_2 - \mathcal{N}_1}{2} O_2^{(1)'}(z_i) + \lambda O_2^{(1)}(z_i)z_i^{-1} \right] \\ &= \sum_{i=1}^M \left[ \frac{\mathcal{N}_2 - \mathcal{N}_1}{12} (\mathcal{N}_1 - 2\mathcal{N}_2 + 2(M-1)c) A''(z_i) \right. \\ & \quad \left. + \frac{\mathcal{N}_1 - 2\mathcal{N}_2 + (M-1)c}{2} \lambda A'(z_i)z_i^{-1} + \lambda(\mathcal{N}_2 - \lambda)A(z_i)z_i^{-2} \right] \\ & \quad - c \sum_{i \neq j}^M \left[ \frac{\mathcal{N}_2 - \mathcal{N}_1}{2} A'(z_i) + 2\lambda A(z_i)z_i^{-1} \right] \frac{1}{z_i - z_j} + \text{const.} \quad (5.14) \end{aligned}$$

On the other hand, we note that the l.h.s. of Eq. (5.12) is a polynomial of at most second-degree. Hence in the polynomial  $A(z)$  of at most fourth-degree defined by Eq. (5.11), the coefficients of the fourth- and zeroth-degree must vanish in order that Eq. (5.12) is met. We can thus put

$$A(z) = a_3 z^3 + a_2 z^2 + a_1 z, \quad O_2^{(1)}(z) = b_2^{(1)} z^2 + b_1^{(1)} z + b_0^{(1)}. \quad (5.15)$$

Substituting the ansatz (5.15) in Eq. (5.13), we find the third condition (5.13) is fulfilled if and only if

$$b_2^{(1)} = \frac{\mathcal{N}_1 - 2\mathcal{N}_2 - 2\lambda - (M-1)c}{2} a_3, \quad (5.16a)$$

$$b_0^{(1)} = \frac{\mathcal{N}_1 - 2\lambda + (M-1)c}{2} a_1. \quad (5.16b)$$

Finally from Eqs. (5.9), (5.10), (5.15), and (5.16), we find that the second-order operators  $\tilde{H}$  which preserve the type C space (3.19) and are equivalent to a Schrödinger operator as well must have the following form:

$$\tilde{H} = - \sum_{i=1}^M A(z_i) \frac{\partial^2}{\partial z_i^2} - \sum_{i=1}^M B(z_i) \frac{\partial}{\partial z_i} - 2c \sum_{i \neq j}^M \frac{A(z_i)}{z_i - z_j} \frac{\partial}{\partial z_i} - \mathbf{C}(\sigma(z)), \quad (5.17)$$

where  $A(z)$  and  $B(z)$  are given by

$$A(z) = a_3 z^3 + a_2 z^2 + a_1 z, \quad (5.18)$$

$$\begin{aligned} B(z) = & - (\mathcal{N} + \lambda - 3 + 2(M-1)c) a_3 z^2 \\ & + (b_1^{(1)} - [\mathcal{N}_1 - 2 + (M-1)c] a_2) z - (\lambda - 1) a_1. \end{aligned} \quad (5.19)$$

Furthermore, the function  $\mathbf{C}$  in Eq. (5.17) is calculated as

$$\begin{aligned}\mathbf{C}(\sigma(z)) &= \frac{\mathcal{N}_1 - 1}{12} (\mathcal{N}_1 - 2 + 2(M - 1)c) \sum_{i=1}^M A''(z_i) \\ &\quad - \frac{\mathcal{N}_1 - 1}{2} \sum_{i=1}^M O_2^{(1)}(z_i) - \frac{\mathcal{N}_1 - 1}{2} c \sum_{i \neq j}^M \frac{A'(z_i)}{z_i - z_j} + R_1 \\ &= (\mathcal{N}_1 - 1)(\mathcal{N}_2 + \lambda - 1)a_3 \sum_{i=1}^M z_i + c_0, \end{aligned} \quad (5.20)$$

where the constants  $R_1$  and  $c_0$  are related with each other so that  $c_0$  in Eq. (5.20) exactly coincides with that in Eq. (4.8f) through the following relation:

$$c_0 = \frac{\mathcal{N}_1 - 1}{6} (\mathcal{N}_1 - 2 - (M - 1)c) M a_2 - \frac{\mathcal{N}_1 - 1}{2} M b_1^{(1)} + R_1. \quad (5.21)$$

In fact, it is easily shown that the gauged Hamiltonian (5.17) can be cast into a Schrödinger operator by a gauge transformation

$$\begin{aligned}H &= e^{-\mathcal{W}} \tilde{H} e^{\mathcal{W}} \\ &= -\frac{1}{2} \sum_{i=1}^M \frac{\partial^2}{\partial q_i^2} + \frac{1}{2} \sum_{i=1}^M \left[ \left( \frac{\partial \mathcal{W}}{\partial q_i} \right)^2 - \frac{\partial^2 \mathcal{W}}{\partial q_i^2} \right] - \mathbf{C}(\sigma(z)), \end{aligned} \quad (5.22)$$

if the gauge potential  $\mathcal{W}$  is chosen as

$$\mathcal{W} = -\sum_{i=1}^M \int dz_i \frac{B(z_i)}{2A(z_i)} + \frac{1}{4} \sum_{i=1}^M \ln |A(z_i)| - c \sum_{i < j}^M \ln |z_i - z_j|, \quad (5.23)$$

and the function  $z(q)$  which determines the change of variables satisfies

$$z'(q)^2 = 2A(z(q)) = 2(a_3 z(q)^3 + a_2 z(q)^2 + a_1 z(q)). \quad (5.24)$$

To make the correspondence between the results here and those in Ref. [18] more transparent, let us decompose  $B(z)$  as

$$B(z) = Q(z) - \frac{\mathcal{N} - 2 + 2(M - 1)c}{2} A'(z), \quad (5.25)$$

so that the coefficients of the second- and zeroth-degree in  $Q(z)$  coincide with each other:

$$Q(z) = \frac{\mathcal{N} - 2\lambda + 2(M - 1)c}{2} A(z) z^{-1} - 2\beta z. \quad (5.26)$$

The parameter  $\beta$  in Eq. (5.26) is given by

$$2\beta = \frac{\mathcal{N}_1 - \mathcal{N}_2 - 2\lambda}{2} a_2 - b_1^{(1)}. \quad (5.27)$$

Then, if we put  $b_1$  as the coefficient of the first-degree in  $Q(z)$  and take  $M = 1$ , we can easily check that Eqs. (5.17)–(5.26) completely reduce to the corresponding quantities in Ref. [18] as they should.

It is apparent from the construction that the Hamiltonian (5.22) preserves the space  $\mathcal{V}_{\mathcal{N}_1, \mathcal{N}_2; M}^{(C)}$  defined by

$$\mathcal{V}_{\mathcal{N}_1, \mathcal{N}_2; M}^{(C)} = e^{-\mathcal{W}} \tilde{\mathcal{V}}_{\mathcal{N}_1, \mathcal{N}_2; M}^{(C)}. \quad (5.28)$$

Hence, the Hamiltonian  $H$  can be (locally) diagonalized in this finite dimensional space (5.28). We shall thus call the space (5.28) the *solvable sector* of  $H$ . To obtain the explicit form of  $\mathcal{V}_{\mathcal{N}_1, \mathcal{N}_2; M}^{(C)}$ , we first arrange the gauge potential (5.23) in a more tractable form:

$$\begin{aligned} \mathcal{W} = & -\frac{\alpha^-}{2} \sum_{i=1}^M \ln |z_i| - \frac{\mathcal{N} - 1 + 2(M-1)c}{4} \sum_{i=1}^M \ln \left| \frac{z_i}{A(z_i)} \right| \\ & + \beta \sum_{i=1}^M \int dz_i \frac{z_i}{A(z_i)} - c \sum_{i < j}^M \ln |z_i - z_j|, \end{aligned} \quad (5.29)$$

where a new parameter  $\alpha^-$  is introduced by

$$\alpha^- = \frac{1 - 2\lambda}{2}. \quad (5.30)$$

Then, from Eqs. (3.19), (5.28), and (5.29), we have

$$\begin{aligned} \mathcal{V}_{\mathcal{N}_1, \mathcal{N}_2; M}^{(C)} = & e^{-\mathcal{U}} \sigma_M(z)^{\frac{1}{2}\alpha^-} \tilde{\mathcal{V}}_{\mathcal{N}_1; M}^{(A)}[z] \oplus e^{-\mathcal{U}} \sigma_M(z)^{\frac{1}{2}(1-\alpha^-)} \tilde{\mathcal{V}}_{\mathcal{N}_2; M}^{(A)}[z] \Big|_{z=z(q)} \\ \equiv & \mathcal{V}_{\mathcal{N}_1; M}^{(1)} \oplus \mathcal{V}_{\mathcal{N}_2; M}^{(2)}, \end{aligned} \quad (5.31)$$

where the new gauge factor  $e^{-\mathcal{U}}$  is defined by

$$\begin{aligned} e^{-\mathcal{U}} = & \prod_{i < j}^M |z_i - z_j|^c \prod_{i=1}^M \left| \frac{z_i}{A(z_i)} \right|^{\frac{1}{4}(\mathcal{N}-1+2(M-1)c)} \\ & \times \exp \left( -\beta \sum_{i=1}^M \int dz_i \frac{z_i}{A(z_i)} \right). \end{aligned} \quad (5.32)$$

## 6 Gauged Hamiltonians in the general type C operators

In the preceding section, we have constructed the type C gauged Hamiltonian  $\tilde{H}$  (5.17) almost independently of the explicit form of the most general type C operator  $\tilde{\mathcal{H}}$  given by Eq. (4.7). In fact, all that we have employed in the previous construction is the known results for the many-body type A gauged

Hamiltonians and the fact that  $\tilde{\mathcal{H}}$  preserves the two type A spaces in Eq. (3.19) separately. In this section, we will thus show how the gauged Hamiltonian  $\tilde{H}$  is embedded in the general type C operator  $\tilde{\mathcal{H}}$ . For this purpose, we shall express  $\tilde{H}$  in terms of  $\sigma_k$ . The necessary formulas are summarized in Appendix.

First from Eqs. (4.8f) and (5.20), we easily see that  $\mathbf{A}_0$  having the form (4.8a) must be

$$\mathbf{A}_0(\sigma) = a_3 \sigma_1. \quad (6.1)$$

With the aid of Eqs. (A.1) and (6.1), the second-order operator in Eq. (5.17) is expressed in terms of  $\sigma_k$  as

$$-\sum_{i=1}^M A(z_i) \frac{\partial^2}{\partial z_i^2} = -\sum_{k,l=1}^M [\mathbf{A}_0(\sigma) \sigma_k \sigma_l + \mathcal{A}_{kl}(\sigma)] \frac{\partial^2}{\partial \sigma_k \partial \sigma_l}, \quad (6.2)$$

where  $\mathcal{A}_{kl}$  are polynomials of second-degree as the following:

$$\begin{aligned} \mathcal{A}_{kl}(\sigma) = & -a_3 \left[ \sum_{m=0}^1 (k-m+1) \sigma_{k-m+1} \sigma_{l+m} \right. \\ & \left. + \sum_{m=2}^{k+1} (k-l-2m+1) \sigma_{k-m+1} \sigma_{l+m} \right] \\ & + a_2 \left[ k \sigma_k \sigma_l + \sum_{m=1}^k (k-l-2m) \sigma_{k-m} \sigma_{l+m} \right] \\ & - a_1 \sum_{m=0}^{k-1} (k-l-2m-1) \sigma_{k-m-1} \sigma_{l+m}. \end{aligned} \quad (6.3)$$

Comparing Eq. (6.2) with the second-order operators in Eq. (4.7), we immediately obtain

$$\mathbf{A}_{\bar{k}\bar{l}}(\sigma) = \mathcal{A}_{\bar{k}\bar{l}}(\sigma) + \mathcal{A}_{\bar{l}\bar{k}}(\sigma) \quad (\bar{k} > \bar{l}), \quad (6.4)$$

$$\mathbf{A}_{\bar{k}\bar{k}}(\sigma) = \mathcal{A}_{\bar{k}\bar{k}}(\sigma), \quad (6.5)$$

$$\mathbf{A}_{M\bar{k}}(\sigma) \sigma_M = \mathcal{A}_{\bar{k}M}(\sigma) + \mathcal{A}_{M\bar{k}}(\sigma), \quad (6.6)$$

$$\mathbf{A}_{MM}(\sigma) \sigma_M = \mathcal{A}_{MM}(\sigma). \quad (6.7)$$

The r.h.s. of Eqs. (6.4) and (6.5) indeed have the same forms as Eqs. (4.8b) and thus completely determine all the polynomials  $\mathbf{A}_{\bar{k}\bar{l}}$  ( $\bar{k} \geq \bar{l}$ ). To factor out  $\sigma_M$  in the r.h.s. of Eq. (6.6), the following formula is useful:

$$\sum_{m=0}^{M-k} (M-k-2m-1) \sigma_{M-m-1} \sigma_{k+m} = (k-M-1) \sigma_{k-1} \sigma_M. \quad (6.8)$$

From Eqs. (6.3)–(6.8), the first-degree polynomials  $\mathbf{A}_{M\bar{k}}$  of the form (4.8c) read

$$\mathbf{A}_{M\bar{k}}(\sigma) = -2(\bar{k}+1)a_3\sigma_{\bar{k}+1} + 2\bar{k}a_2\sigma_{\bar{k}} - 2(\bar{k}-M-1)a_1\sigma_{\bar{k}-1}, \quad (6.9)$$

$$\mathbf{A}_{MM}(\sigma) = Ma_2\sigma_M + a_1\sigma_{M-1}. \quad (6.10)$$

Similarly, with the aid of the formulas (A.2), (A.3) and Eq. (6.1) the first-order operators in Eq. (5.17) are expressed in terms of  $\sigma_k$  as

$$\begin{aligned} -\sum_{i=1}^M B(z_i) \frac{\partial}{\partial z_i} - 2c \sum_{i \neq j}^M \frac{A(z_i)}{z_i - z_j} \frac{\partial}{\partial z_i} \\ = \sum_{k=1}^M \left[ (\mathcal{N} + \lambda - 3) \mathbf{A}_0(\sigma) - \mathcal{B}_k(\sigma) \right] \frac{\partial}{\partial \sigma_k}, \end{aligned} \quad (6.11)$$

where  $\mathcal{B}_k$  are polynomials of first-degree as the following:

$$\begin{aligned} \mathcal{B}_k(\sigma) = (k+1)(\mathcal{N} + \lambda - 3 + kc)a_3\sigma_{k+1} \\ - \frac{k}{2} \left( [\mathcal{N} + 2\lambda - 4 + 2(k-M)c]a_2 + 4\beta \right) \sigma_k \\ + (k-M-1)(\lambda - 1 + (k-M)c)a_1\sigma_{k-1}. \end{aligned} \quad (6.12)$$

In particular,  $\mathcal{B}_M$  can be arranged with the use of Eq. (6.10) as

$$\mathcal{B}_M(\sigma) = -(\lambda - 1)\mathbf{A}_{MM}(\sigma) - \frac{M}{2} \left( (\mathcal{N} - 2)a_2 + 4\beta \right) \sigma_M. \quad (6.13)$$

It is now apparent that  $\mathcal{B}_k$  are all in agreement with the form of  $\mathbf{B}_k$  given by Eqs. (4.8d) and (4.8e) and thus

$$\mathbf{B}_k(\sigma) = \mathcal{B}_k(\sigma). \quad (6.14)$$

It should be pointed out that after determining all the parameters  $A_{ik,jl}$  from Eqs. (6.4), (6.5), (6.9), and (6.10) we can always adjust the parameters  $B_{i\bar{k}}$  and  $B_{0\bar{k}}$  in Eq. (4.8d) and  $B_{MM}$  in Eq. (4.8e) so that Eq. (6.14) holds. Therefore, the type C gauged Hamiltonian (5.17) in fact constitutes a subclass of the general type C operator (4.7).

From Eq. (6.1) and the discussion following Eq. (4.8f), it is apparent that the gauged Hamiltonian  $\tilde{H}$  is solvable if

$$a_3 = 0. \quad (6.15)$$

In this case, the polynomial  $\mathbf{C}$  given by Eq. (5.20) is a constant and hence the many-body Hamiltonian (5.22) is of the supersymmetric form up to this irrelevant constant. The gauge potential  $\mathcal{W}$  can be regarded as a superpotential.



## 7 Classification of the Models

We shall now explicitly compute the type C models. The potential term in Eq. (5.22) is explicitly calculated in terms of  $z_i$  from Eqs. (5.18)–(5.23) as

$$\begin{aligned}
V = & \sum_{i=1}^M \frac{1}{4A(z_i)} \left[ \frac{\mathcal{N} - 1 + 2(M-1)c}{2} A'(z_i) - Q(z_i) \right] \\
& \times \left[ \frac{\mathcal{N} + 1 + 2(M-1)c}{2} A'(z_i) - Q(z_i) \right] - a_3(M, \mathcal{N}_i, \lambda) \sum_{i=1}^M z_i \\
& + c(c-1) \sum_{i < j}^M \frac{A(z_i) + A(z_j)}{(z_i - z_j)^2} + V_0, \tag{7.1}
\end{aligned}$$

where, and in what follows,  $V_0$  denotes an arbitrary constant, and the coupling constant  $a_3(M, \mathcal{N}_i, \lambda)$  is given by

$$\begin{aligned}
a_3(M, \mathcal{N}_i, \lambda) = & \frac{1}{2} \left[ 2\mathcal{N}_1(\mathcal{N}_2 + \lambda) + 2(\mathcal{N} + \lambda)(M-1)c \right. \\
& \left. + 2M(M-1)c^2 - 2(M-1)c - 1 \right] a_3. \tag{7.2}
\end{aligned}$$

From Eq. (5.24), the change of variable is determined by the elliptic integral:

$$\pm(q - q_0) = \int \frac{dz}{\sqrt{2(a_3 z^3 + a_2 z^2 + a_1 z)}}. \tag{7.3}$$

In contrast to the systems constructed from the  $\mathfrak{sl}(M+1)$  generators, our present models do not have full  $GL(2, \mathbb{R})$  invariance; the conditions (5.12) and (5.13) apparently break the  $GL(2, \mathbb{R})$  symmetry. However, there remains residual symmetry unbroken, namely, the invariance under scale transformations,  $\beta = \gamma = 0$  in Eq. (5.1), and under special projective transformations,  $\alpha = \delta = 0$  and  $\beta = \gamma (\neq 0)$  in Eq. (5.1). The latter invariance is as it should be since we have made the generalization so that Eq. (3.17) holds. Due to the invariance, we can classify the models completely the same way as in the single-variable case presented in Ref. [18]. This is another virtue of our choice of the space (3.19) based on the invariance criterion. It is readily shown that  $A(z)$  can be cast into one of the canonical forms listed in Table 1 by the combination of the aforementioned scale and projective transformations. For the detailed treatment of the elliptic integral (7.3) in each case, see Ref. [18].

Furthermore, we note that from Eq. (5.24), a rescaling of the coefficients  $a_i$ ,  $\beta$ ,  $c_0$  by an overall constant factor  $\nu > 0$  has the following effect on the change of variable  $z(q)$ :

$$z(q; \nu a_i, \nu \beta, \nu c_0) = z(\sqrt{\nu} q; a_i, \beta, c_0). \tag{7.4}$$

Table 1

Canonical forms for the polynomial  $A(z)$ , (5.18).

| Case | Canonical Form                         |
|------|--|
| 1    | $2z$                                   |
| 2    | $\pm 2\nu z^2$                         |
| 3    | $\pm 2\nu z(1-z)(1-mz)$                |
| 4    | $2\nu z(1-z)(m' + mz)$                 |
| 5    | $\frac{1}{2}\nu z(z^2 + 2(1-2m)z + 1)$ |

In this Table,  $\nu > 0$ ,  $0 \leq m \leq 1$  and  $m' = 1 - m$  ( $m \neq 1$  in Case 3 and  $m \neq 0, 1$  in Case 4 to avoid duplications).

From this equation and Eqs. (5.18)–(5.20), (5.22), and (5.23) we easily obtain the identities

$$\mathcal{W}(q; \nu a_i, \nu \beta, \nu c_0) = \mathcal{W}(\sqrt{\nu} q; a_i, \beta, c_0) \quad (7.5a)$$

$$V(q; \nu a_i, \nu \beta, \nu c_0) = \nu V(\sqrt{\nu} q; a_i, \beta, c_0). \quad (7.5b)$$

We shall therefore set  $\nu = 1$  in the canonical forms in Cases 2–5, the models corresponding to an arbitrary value of  $\nu$  following easily from Eqs. (7.4) and (7.5). It should also be obvious from Eq. (7.3) that the change of variable  $z(q)$ , and hence the potential  $V$  determining each model, are defined up to the transformation  $q \mapsto \pm(q - q_0)$ , where  $q_0 \in \mathbb{R}$  is a constant. In the following, another new parameter  $\alpha^+$  is introduced by

$$\alpha^+ = \mathcal{N}_2 - \mathcal{N}_1 + \lambda + \frac{1}{2}. \quad (7.6)$$

The condition for solvability (6.15) implies that the models corresponding to the first and second canonical forms in Table 1, and to the third one with  $m = 0$ , are not only quasi-solvable but also solvable.

As we will see below, the potentials  $V$  in all the cases but the irrelevant Case 2 have the singularities at  $q_i = 0$  and at integer multiples of a real number  $\Omega$ ,  $q_i = n\Omega$  ( $n \in \mathbb{Z}$ ), where  $\Omega$  is a submultiple of the real period of the potential  $V$  or  $\Omega = \infty$  when the potential is non-periodic, as well as  $q_i = \pm q_j$  for  $i \neq j$ . Hence, the Hamiltonians are naturally defined on

$$0 < q_M < \cdots < q_1 < \Omega. \quad (7.7)$$

The potential form of the one-body part in each case is completely identical with that of one-body type C  $\mathcal{N}$ -fold supersymmetric potential  $V^-$  in Ref. [18]. Thus, the normalizability conditions coming from the one-body part, which restrict the values of the parameters  $\alpha^\pm$  and  $\beta^\pm$  in each case, are completely

the same as those shown in this reference. Only an additional condition arises from the two-body part. The finiteness of the  $L^2$  norm of the two-body wave function in the solvable sector  $\mathcal{V}_{\mathcal{N}_1, \mathcal{N}_2; M}^{(C)}$  leads  $c > -1/2$ , where  $c$  denotes the coupling constant of the two-body interaction appeared in the last line of Eq. (7.1).

### 7.1 Case 1: $A(z) = 2z$

*Change of variable:*  $z = q^2$ .

*Potential:*

$$V = \frac{\beta^2}{2} \sum_{i=1}^M q_i^2 + \frac{\alpha^-(\alpha^- - 1)}{2} \sum_{i=1}^M \frac{1}{q_i^2} + c(c-1) \sum_{i < j}^M \left[ \frac{1}{(q_i - q_j)^2} + \frac{1}{(q_i + q_j)^2} \right] + V_0. \quad (7.8)$$

*Solvable sectors:*

$$\mathcal{V}_{\mathcal{N}_1; M}^{(1)} = \prod_{i < j}^M |q_i^2 - q_j^2|^c \left( \prod_{i=1}^M q_i^{\alpha^-} \right) \exp \left( -\frac{\beta}{2} \sum_{i=1}^M q_i^2 \right) \tilde{\mathcal{V}}_{\mathcal{N}_1; M}^{(A)}[q^2], \quad (7.9a)$$

$$\mathcal{V}_{\mathcal{N}_2; M}^{(2)} = \prod_{i < j}^M |q_i^2 - q_j^2|^c \left( \prod_{i=1}^M q_i^{1-\alpha^-} \right) \exp \left( -\frac{\beta}{2} \sum_{i=1}^M q_i^2 \right) \tilde{\mathcal{V}}_{\mathcal{N}_2; M}^{(A)}[q^2], \quad (7.9b)$$

where  $\mathcal{V}_{\mathcal{N}_k; M}^{(k)}$  ( $k = 1, 2$ ) are defined in Eq. (5.31). This case corresponds to the rational  $BC_M$  type Calogero-Sutherland model [20]. The one-body part of the potential is only singular at  $q_i = 0$  and hence a natural choice is  $\Omega = \infty$ .

### 7.2 Case 2a: $A(z) = 2z^2$

*Change of variable:*  $z = e^{2q}$ .

*Potential:*

$$V = c(c-1) \sum_{i < j}^M \frac{1}{\sinh^2(q_i - q_j)} + V_0. \quad (7.10)$$

*Solvable sectors:*

$$\mathcal{V}_{\mathcal{N}_1;M}^{(1)} = \prod_{i<j}^M |e^{2q_i} - e^{2q_j}|^c \left( \prod_{i=1}^M e^{(\alpha^- - \bar{\beta})q_i} \right) \tilde{\mathcal{V}}_{\mathcal{N}_1;M}^{(A)}[e^{2q}], \quad (7.11a)$$

$$\mathcal{V}_{\mathcal{N}_2;M}^{(2)} = \prod_{i<j}^M |e^{2q_i} - e^{2q_j}|^c \left( \prod_{i=1}^M e^{(1-\alpha^- - \bar{\beta})q_i} \right) \tilde{\mathcal{V}}_{\mathcal{N}_2;M}^{(A)}[e^{2q}]. \quad (7.11b)$$

*Parameter:*

$$\bar{\beta} = \beta + \frac{\mathcal{N} - 1 + 2(M - 1)c}{2}. \quad (7.12)$$

We note that the potential (7.10) depends on neither the parameter  $\alpha^-$  nor  $\bar{\beta}$  appeared in the solvable sectors (7.11). The dependence on these parameters only comes from the middle part of the each solvable sector. These parts are nothing but the wave functions corresponding to the free motion of the center-of-mass due to the translational invariance of the potential (7.10) and thus irrelevant. The remaining factors in Eqs. (7.11a) and (7.11b) coincide with each other and the dependence on  $\mathcal{N}_k$  is not relevant since this case is solvable. Therefore, Case 2 only reproduces the well-known result of the (hyperbolic)  $A_{M-1}$  Sutherland model [8].

### 7.3 Case 2b: $A(z) = -2z^2$

The formula for the potential for this case can be easily deduced from those of the preceding one by applying Eqs. (7.5) with  $\nu = -1$ . Apparently, this case only shows the usual solvability of the trigonometric  $A_{M-1}$  Sutherland model.

### 7.4 Case 3a: $A(z) = 2z(1 - z)(1 - mz)$

*Change of variable:*  $z = \text{sn}^2 q$ .

Here, and in the following cases, the Jacobi elliptic functions have modulus  $k = \sqrt{m}$ .

*Potential:*

$$V = \frac{m\alpha^+(\alpha^+ - 1)}{2} \sum_{i=1}^M \text{sn}^2 q_i + \frac{\alpha^-(\alpha^- - 1)}{2} \sum_{i=1}^M \frac{1}{\text{sn}^2 q_i}$$

$$\begin{aligned}
& + \frac{m'\beta^-(\beta^- - 1)}{2} \sum_{i=1}^M \frac{1}{\text{cn}^2 q_i} - \frac{m'\beta^+(\beta^+ - 1)}{2} \sum_{i=1}^M \frac{1}{\text{dn}^2 q_i} \\
& + c(c-1) \sum_{i<j}^M \left[ \frac{1}{\text{sn}^2(q_i - q_j)} + \frac{1}{\text{sn}^2(q_i + q_j)} \right] + V_0. \quad (7.13)
\end{aligned}$$

*Parameters:*

$$\beta^\pm = \mp \frac{\beta}{m'} - \frac{1}{2}(\mathcal{N} - 1 + 2(M-1)c). \quad (7.14)$$

*Solvable sectors:*

$$\begin{aligned}
\mathcal{V}_{\mathcal{N}_1;M}^{(1)} &= \prod_{i<j}^M |\text{sn}^2 q_i - \text{sn}^2 q_j|^c \\
&\times \prod_{i=1}^M [(\text{sn } q_i)^{\alpha^-} (\text{cn } q_i)^{\beta^-} (\text{dn } q_i)^{\beta^+}] \tilde{\mathcal{V}}_{\mathcal{N}_1;M}^{(A)}[\text{sn}^2 q], \quad (7.15a)
\end{aligned}$$

$$\begin{aligned}
\mathcal{V}_{\mathcal{N}_2;M}^{(2)} &= \prod_{i<j}^M |\text{sn}^2 q_i - \text{sn}^2 q_j|^c \\
&\times \prod_{i=1}^M [(\text{sn } q_i)^{1-\alpha^-} (\text{cn } q_i)^{\beta^-} (\text{dn } q_i)^{\beta^+}] \tilde{\mathcal{V}}_{\mathcal{N}_2;M}^{(A)}[\text{sn}^2 q]. \quad (7.15b)
\end{aligned}$$

We note that from the formula

$$\frac{m'}{\text{cn}^2 q} = -m \text{sn}^2 q - \frac{1}{\text{sn}^2 q} + \frac{4}{\text{sn}^2 2q} + \frac{m'}{\text{dn}^2 q} - (1+m), \quad (7.16)$$

we can regard the above model (7.13) as a deformation of an Olshanetsky–Perelomov type potential associated with the  $BC_M$  root system [20]. In contrast to the ordinary elliptic integrable models, the potential function here is given by the Jacobian function instead of the Weierstrass  $\wp$  function. We thus call the potential (7.13) *Jacobi-elliptic  $BC_M$  Inozemtsev model*. The one-body part of the potential have real singularities at the points  $q_i = nK(m)$  ( $n \in \mathbb{Z}$ ) where  $K(m)$  is the complete elliptic integral of the first kind defined by

$$K(m) \equiv \int_0^{\pi/2} \frac{dt}{\sqrt{1 - m \sin^2 t}}. \quad (7.17)$$

Hence a natural choice is  $\Omega = K(m)$ .

As previously mentioned, when  $m = 0$  the polynomial  $A(z)$  is of second degree and hence the Hamiltonian  $H$  is solvable. The formulas for the potential and the solvable sectors are obtained from Eqs. (7.13)–(7.15) by setting  $m = 0$ ,  $m' = 1$ , and  $(\text{sn } q, \text{cn } q, \text{dn } q) = (\sin q, \cos q, 1)$ . With the aid of the formula (7.16), we immediately see that the potential (7.13) in this case is nothing but the trigonometric  $BC_M$  Calogero–Sutherland model.

### 7.5 Case 3b: $A(z) = -2z(1-z)(1-mz)$

As in Case 2b, the formulas for this case can follow from those of the preceding one using Eqs. (7.5) with  $\nu = -1$ . The following well-known identities [21] may be of help in this case:

$$\operatorname{sn}(iq; m) = i \frac{\operatorname{sn}(q; m')}{\operatorname{cn}(q; m')}, \quad \operatorname{cn}(iq; m) = \frac{1}{\operatorname{cn}(q; m')}, \quad \operatorname{dn}(iq; m) = \frac{\operatorname{dn}(q; m')}{\operatorname{cn}(q; m')}.$$

The real singularities of the one-body part of the potential are now at  $q_i = nK(m')$  ( $n \in \mathbb{Z}$ ) and thus we naturally choose as  $\Omega = K(m')$ . The value  $m = 0$  yields again solvable model, whose potentials and solvable sectors follow from Eqs. (7.13)–(7.15) by setting  $m = 0$ ,  $m' = 1$ , and  $(\operatorname{sn} iq, \operatorname{cn} iq, \operatorname{dn} iq) = (i \sinh q, \cosh q, 1)$ . Clearly, the potential in this case corresponds to the hyperbolic  $BC_M$  Calogero-Sutherland model.

### 7.6 Case 4: $A(z) = 2z(1-z)(m' + mz)$

*Change of variable:*  $z = \operatorname{cn}^2 q$ .

*Potential:*

$$\begin{aligned} V = & \frac{m\alpha^+(\alpha^+ - 1)}{2} \sum_{i=1}^M \operatorname{sn}^2 q_i + \frac{m'\alpha^-(\alpha^- - 1)}{2} \sum_{i=1}^M \frac{1}{\operatorname{cn}^2 q_i} \\ & + \frac{\beta^-(\beta^- - 1)}{2} \sum_{i=1}^M \frac{1}{\operatorname{sn}^2 q_i} - \frac{m'\beta^+(\beta^+ - 1)}{2} \sum_{i=1}^M \frac{1}{\operatorname{dn}^2 q_i} \\ & + c(c-1) \sum_{i < j}^M \left[ \frac{1}{\operatorname{sn}^2(q_i - q_j)} + \frac{1}{\operatorname{sn}^2(q_i + q_j)} \right] + V_0. \end{aligned} \quad (7.18)$$

*Parameters:*

$$\beta^\pm = \mp \beta - \frac{1}{2}(\mathcal{N} - 1 + 2(M-1)c). \quad (7.19)$$

*Solvable sectors:*

$$\begin{aligned} \mathcal{V}_{\mathcal{N}_1; M}^{(1)} &= \prod_{i=1}^M \left| \operatorname{sn}^2 q_i - \operatorname{sn}^2 q_j \right|^c \\ &\quad \times \prod_{i=1}^M \left[ (\operatorname{cn} q_i)^{\alpha^-} (\operatorname{sn} q_i)^{\beta^-} (\operatorname{dn} q_i)^{\beta^+} \right] \tilde{\mathcal{V}}_{\mathcal{N}_1; M}^{(A)} [\operatorname{cn}^2 q], \end{aligned} \quad (7.20a)$$

$$\mathcal{V}_{\mathcal{N}_2; M}^{(2)} = \prod_{i=1}^M \left| \operatorname{sn}^2 q_i - \operatorname{sn}^2 q_j \right|^c$$

$$\times \prod_{i=1}^M \left[ (\text{cn } q_i)^{1-\alpha^-} (\text{sn } q_i)^{\beta^-} (\text{dn } q_i)^{\beta^+} \right] \tilde{\mathcal{V}}_{\mathcal{N}_2;M}^{(A)}[\text{cn}^2 q]. \quad (7.20b)$$

We again obtain a Jacobi-elliptic  $BC_M$  Inozemtsev model. The location of the real singularities are completely the same as that in Case 3a.

7.7 Case 5:  $A(z) = z(z^2 + 2(1 - 2m)z + 1)/2$

Change of variable:  $z = (1 + \text{cn } q)/(1 - \text{cn } q)$ .

Potential:

$$\begin{aligned} V = & \frac{\alpha^+(\alpha^+ - 1)}{4} \sum_{i=1}^M \frac{1}{1 - \text{cn } q_i} + \frac{\alpha^-(\alpha^- - 1)}{4} \sum_{i=1}^M \frac{1}{1 + \text{cn } q_i} \\ & + \left[ \frac{\beta^2}{2m} - \frac{m'}{8} (\mathcal{N} - 1 + 2(M - 1)c) (\mathcal{N} + 1 + 2(M - 1)c) \right] \sum_{i=1}^M \frac{1}{\text{dn}^2 q_i} \\ & + (\mathcal{N} + 2(M - 1)c) \frac{\beta}{2} \sum_{i=1}^M \frac{\text{cn } q_i}{\text{dn}^2 q_i} \\ & + c(c - 1) \sum_{i < j}^M \left[ \frac{(1 - \text{cn } q_i \text{cn } q_j)(m' + m \text{cn } q_i \text{cn } q_j)}{(\text{cn } q_i - \text{cn } q_j)^2} \right] + V_0. \quad (7.21) \end{aligned}$$

Gauge factor:

$$\begin{aligned} e^{-\mathcal{U}^\pm} = & \prod_{i < j}^M \left| \frac{1 + \text{cn } q_i}{1 - \text{cn } q_i} - \frac{1 + \text{cn } q_j}{1 - \text{cn } q_j} \right|^c \prod_{i=1}^M \left( \frac{1 - \text{cn } q_i}{\text{dn } q_i} \right)^{\frac{1}{2}(\mathcal{N} - 1 + 2(M - 1)c)} \\ & \times \exp \left( -\frac{\beta}{kk'} \sum_{i=1}^M \arctan \frac{k'^2 + k^2 \text{cn } q_i}{kk'(1 - \text{cn } q_i)} \right), \quad (7.22) \end{aligned}$$

where  $k' = \sqrt{m'} = \sqrt{1 - m}$ . The one-body part of the potential is singular at integer multiples of  $2K(m)$ , so that a natural choice is  $\Omega = 2K(m)$ . This model is also regarded as a kind of Jacobi-elliptic  $BC_M$  Inozemtsev system since this case is obtained by a *complex* projective transformation from Case 3( $m \neq 0$ ) or Case 4 (cf. Section 9).

## 8 Difficulty in Type B Generalization

Before concluding the article, we will give brief comments on the several-variable generalization of the type B monomial space and its difficulty. The

type B monomial space of a single variable is defined by

$$\tilde{\mathcal{V}}_{\mathcal{N}}^{(B)} = \langle 1, z, \dots, z^{\mathcal{N}-2}, z^{\mathcal{N}} \rangle. \quad (8.1)$$

The general quasi-solvable operators preserving the space (8.1) and the corresponding  $\mathcal{N}$ -fold supersymmetry are investigated in Ref. [22]. It is pointed out in Ref. [18] that the type B space (8.1) is realized from the type C space (3.6) by the following formal substitution

$$\mathcal{N}_2 = 1, \quad \lambda = \mathcal{N}_1 + 1 = \mathcal{N}, \quad (8.2)$$

although the characteristic features of the type B and C quasi-solvable operators are quite different from each other. Hence, a natural generalization of the type B space is achieved by the same substitution as Eq. (8.2) in the type C space of several variables (3.19), which results in

$$\tilde{\mathcal{V}}_{\mathcal{N};M}^{(B)} = \tilde{\mathcal{V}}_{\mathcal{N}-1;M}^{(A)} \oplus \langle \sigma_M^{\mathcal{N}} \rangle. \quad (8.3)$$

We easily see that the above space (8.3) indeed reduces to Eq. (8.1) when  $M = 1$ . In contrast to the single-variable case, however, there is no second-order operator preserving the space (8.3) which raises the degree of the monomials by two; such an operator must delete *simultaneously* all the elements  $\sigma_1^{n_1} \dots \sigma_M^{n_M}$  with  $\sum_i n_i = \mathcal{N} - 3$  and  $\mathcal{N} - 2$  as well as the element  $\sigma_M^{\mathcal{N}}$ , but it cannot be achieved by any at most second-order operator. In other words, there is no several-variable counterpart of an operator which reduces (when  $M = 1$ ) to the degree two second-order operator  $J_{++}$  preserving the single-variable type B space (8.1):

$$J_{++} = z^2 \left( z \frac{d}{dz} - \mathcal{N} \right) \left( z \frac{d}{dz} - \mathcal{N} + 3 \right). \quad (8.4)$$

Needless to say, there are many other possibilities for the generalizations instead of Eq. (8.3). However, this kind of difficulty may not be the sole obstacle and, as far as we have investigated so far, no natural generalization of the single-variable type B has been found.

## 9 Discussion and Summary

In this article, we have developed a systematic procedure to construct a family of quasi-solvable operators in several variables leaving two type A spaces separately and thus admitting two different polynomial type solutions. It is evident that this family is regarded as a subclass of the operators which leave a single type A space (3.10) invariant and thus ensure only one polynomial type solution. The latter operators are exactly those constructed from the



Table 2

Correspondence between type A and type C canonical forms

| Type A |                         | $GL(2, \mathbb{C})$  | Type C              |
|--------|-------------------------|--|---------------------|
| Case   | Canonical Form          | $\longrightarrow$  | Case                |
| I      | $1/2$                   | —  | —                   |
| II     | $2z$                    | Trivial  | 1                   |
| III    | $2z^2$                  | Trivial  | 2                   |
| IV     | $2(z^2 - 1)$            | $\alpha = -2\beta = -2\delta$<br>$= -\sqrt{2}e^{i\pi/4}, \gamma = 0$ | $3(m = 0)$          |
| V      | $2z^3 - g_2z/2 - g_3/2$ | $\beta = e_1, \gamma = 0, \delta = 1$                                | $3(m \neq 0), 4, 5$ |

In this Table,  $g_2, g_3 \in \mathbb{C}$  satisfy  $g_2^3 - 27g_3^2 \neq 0$ , and  $e_1 \in \mathbb{C}$  is one of the three different roots  $e_i$  ( $i = 1, 2, 3$ ) of  $4\beta^3 - g_2\beta - g_3 = 0$ .

full  $\mathfrak{sl}(M + 1)$  generators. Therefore, each Hamiltonian of the five cases classified here must fall within one of the classes of the  $\mathfrak{sl}(M + 1)$  Lie-algebraic Hamiltonians classified in Ref. [13]. To see the correspondence, it is convenient to extend the free parameters in the Hamiltonians to be complex-valued and to consider the  $GL(2, \mathbb{C})$  of complex linear fractional transformations in Eq. (5.1).

In Table 2, the first and second columns show the classification scheme and the canonical form of the polynomials  $A(z)$  under  $GL(2, \mathbb{C})$ , respectively, for the  $\mathfrak{sl}(M + 1)$  Lie-algebraic Hamiltonians preserving one type A space, the third column shows the concrete set of the parameters of the  $GL(2, \mathbb{C})$  transformation which renders the each canonical form of the type A to one of the canonical form of the type C, and the fourth column shows the corresponding type C case classified in this article. For example, we can read from Table 2 that Case 3 with  $m = 0$  of the type C is equivalent to Case IV of the type A under the  $GL(2, \mathbb{C})$  transformation while all of Case 3( $m \neq 0$ ), 4, and 5 are to Case V.

Combining the results here with those in Ref. [13], we finally obtain Table 3. In Table 3, the number 1 or 2 indicates the number of different families of polynomial type solutions available for each model while the symbol  $\times$  means that the corresponding model is only quasi-solvable but not solvable. It is interesting to note that quantum (quasi-)solvability favors the integrable models associated with the  $BC$  root system over those associated with the  $A$  root system. In the construction of the type C models, on the other hand, we have not assumed *a priori* the invariance under the Weyl group of the  $BC$  root system. In this respect, we have not appreciated whether this unbalance comes from

Table 3  
Classification of the (quasi-)solvable quantum many-body systems

| • Class (• Subclass)  | Quasi-solvable | Solvable |
|---|----------------|----------|
| • Rational $A$ Inozemtsev   | 1              | ×        |
| • Rational $A$ Calogero-Sutherland                                  | 1              | 1        |
| • Rational $BC$ Inozemtsev  | 1              | ×        |
| • Rational $BC$ Calogero-Sutherland                                 | 2              | 2        |
| • Hyp.(Trig.) $A$ Inozemtsev  | 1              | ×        |
| • Hyp.(Trig.) $A$ Calogero-Sutherland<br>+ external Morse potential | 1              | 1        |
| • Hyp.(Trig.) $BC$ Inozemtsev                                       | 1              | ×        |
| • Hyp.(Trig.) $BC$ Calogero-Sutherland                              | 2              | 2        |
| • Elliptic $BC$ Inozemtsev  | 1              | ×        |
| • Elliptic $BC$ Calogero-Sutherland                                 | 1              | ×        |
| • Jacobi-elliptic $BC$ Inozemtsev                                   | 2              | ×        |

a deeper mathematical structure.

As was discussed in the previous paper [18], one of the two different families of polynomial type solutions in most cases does not satisfy the normalizability on a space where the Hamiltonian can be naturally defined and thus does not provide a physical state. Even in the case where both of them can satisfy the normalizability on this space, one of them is still weakly singular at  $q_i = 0$  and thus it is natural to discard it as unphysical. As far as the author's knowledge, however, there has been no general principle in quantum mechanics which enforces us to do it<sup>1</sup>. Indeed, some recent investigations have rather indicated the significance of the careful treatment of this kind of solution, cf. Refs. [24,25] and references cited therein. In particular, it was shown in Ref. [24] that by employing an ingenious boundary condition at the singularity of the potential proper linear combinations of the two linearly independent normalizable solutions, which still have a weak singularity, may be well-defined as physical eigenstates of the Hamiltonian. Therefore, the possibility of realizing such a novel physical state would deserve further investigations both theoretical and experimental.

Although we have mainly concentrated on the construction of quantum many-body Hamiltonians in this article, the procedure is quite general and may

<sup>1</sup> One of the most acceptable postulates could be that in Ref. [23].

have wider applicability. We may construct, for instance, a higher-order linear differential operator which admits two or more linearly independent series or polynomial type solutions by a slight modification of the present approach.

Finally, it should be pointed out that the manipulation employed in Section 5 is justified only after we recognize that the most general type C quasi-solvable operator preserves the two subspaces in Eq. (3.19) separately. In other words, an analogous condition to Eq. (5.9) in general provides only a sufficient condition for an operator to preserve the type C space. In fact, we have found that, without the restriction on the value of the parameter  $\lambda$  given by Eq. (3.7), the most general type C operator no longer preserves the two subspaces separately. The restriction (3.7) was originally introduced [18] in order to prevent the type C space (3.6) of a single variable from reducing to the type A space (3.9). In the several-variable case, however, it is easily seen that the generalized type C space (3.19) does not reduce to the type A space (3.10) of several variables even if the parameter  $\lambda$  takes a forbidden value in Eq. (3.7). Therefore, there exists the cases, which we shall refer to as *irregular cases*, where the monomial space is of the type C and different from the type A but nevertheless the general quasi-solvable operator for it does not preserve the two subspaces separately. These irregular cases need separate treatment from the regular ones presented in this article and will be reported elsewhere.

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## A Formulas

In this appendix, we summarize the formulas connecting the differential operators in terms of  $z_i$  with those in terms of  $\sigma_k$ . For the details of the derivation, see Ref. [13].

$$\sum_{i=1}^M z_i \frac{\partial^2}{\partial z_i^2} = - \sum_{k,l=1}^M \left[ \sum_{m=0}^{k-1} (k-l-2m-1) \sigma_{k-m-1} \sigma_{l+m} \right] \frac{\partial^2}{\partial \sigma_k \partial \sigma_l}, \quad (\text{A.1a})$$

$$\sum_{i=1}^M z_i^2 \frac{\partial^2}{\partial z_i^2} = \sum_{k,l=1}^M \left[ k \sigma_k \sigma_l + \sum_{m=1}^k (k-l-2m) \sigma_{k-m} \sigma_{l+m} \right] \frac{\partial^2}{\partial \sigma_k \partial \sigma_l}, \quad (\text{A.1b})$$

$$\sum_{i=1}^M z_i^3 \frac{\partial^2}{\partial z_i^2} = \sum_{k,l=1}^M \left[ \sigma_1 \sigma_k \sigma_l - \sum_{m=0}^1 (k-m+1) \sigma_{k-m+1} \sigma_{l+m} - \sum_{m=2}^{k+1} (k-l-2m+1) \sigma_{k-m+1} \sigma_{l+m} \right] \frac{\partial^2}{\partial \sigma_k \partial \sigma_l}. \quad (\text{A.1c})$$

$$\sum_{i=1}^M \frac{\partial}{\partial z_i} = - \sum_{k=1}^M (k-M-1) \sigma_{k-1} \frac{\partial}{\partial \sigma_k}, \quad (\text{A.2a})$$

$$\sum_{i=1}^M z_i \frac{\partial}{\partial z_i} = \sum_{k=1}^M k \sigma_k \frac{\partial}{\partial \sigma_k}, \quad (\text{A.2b})$$

$$\sum_{i=1}^M z_i^2 \frac{\partial}{\partial z_i} = \sum_{k=1}^M \left[ \sigma_1 \sigma_k - (k+1) \sigma_{k+1} \right] \frac{\partial}{\partial \sigma_k}. \quad (\text{A.2c})$$

$$2 \sum_{i \neq j}^M \frac{z_i}{z_i - z_j} \frac{\partial}{\partial z_i} = \sum_{k=1}^M (k-M)(k-M-1) \sigma_{k-1} \frac{\partial}{\partial \sigma_k}, \quad (\text{A.3a})$$

$$2 \sum_{i \neq j}^M \frac{z_i^2}{z_i - z_j} \frac{\partial}{\partial z_i} = - \sum_{k=1}^M k(k-2M+1) \sigma_k \frac{\partial}{\partial \sigma_k}, \quad (\text{A.3b})$$

$$2 \sum_{i \neq j}^M \frac{z_i^3}{z_i - z_j} \frac{\partial}{\partial z_i} = \sum_{k=1}^M \left[ 2(M-1) \sigma_1 \sigma_k + (k+1)(k-2M+2) \sigma_{k+1} \right] \frac{\partial}{\partial \sigma_k}. \quad (\text{A.3c})$$

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